Convolution identities for Cauchy numbers of the first kind and of the second kind (Analytic Number Theory : Arithmetic Properties of Transcendental Functions and their Applications)

Komatsu, Takao

数理解析研究所講究録
第 1898 巻 2014 年 174-193

Convolution identities for Cauchy numbers of the first kind and of the second kind

Takao Komatsu [1]
Graduate School of Science and Technology
Hirosaki University

1 Introduction

The Cauchy numbers of the first kind c_n $(n \geq 0)$ are defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

and the generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). The first few values are

$$c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{6},\ c_3 = \frac{1}{4},\ c_4 = -\frac{19}{30},\ c_5 = \frac{9}{4},\ c_6 = -\frac{863}{84}.$$

The Cauchy numbers are almost identical with the Bernoulli numbers of the second kind, b_n, as introduced in [8] and later studied by various authors. In fact, we have $c_n = n! b_n$. Notice that the Cauchy numbers c_n can be expressed in terms of the unsigned Stirling numbers of the first kind and $\left[{n \atop i}\right]$:

$$c_n = \sum_{i=0}^{n} \frac{(-1)^{n-i}}{i+1} \left[{n \atop i}\right] \tag{1}$$

[1] This research was supported in part by the Grant-in-Aid for Scientific research (C) (No.22540005), the Japan Society for the Promotion of Science.

([9, 11]), where the unsigned Stirling numbers of the first kind $\left[{n \atop i}\right]$ arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{i=0}^{n} \left[{n \atop i}\right] x^i$$

(see e.g. [7]).

The Cauchy numbers of the second kind $\widehat{c}_n$ $(n \geq 0)$ are defined by

$$\widehat{c}_n = \int_0^1 (-x)(-x-1)\dots(-x-n+1)dx$$

and the generating function of $\widehat{c}_n$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). The first few values are

$$\widehat{c}_0 = 1,\ \widehat{c}_1 = -\frac{1}{2},\ \widehat{c}_2 = \frac{5}{6},\ \widehat{c}_3 = -\frac{9}{4},\ \widehat{c}_4 = \frac{251}{30},\ \widehat{c}_5 = -\frac{475}{12},\ \widehat{c}_6 = \frac{19087}{84}.$$

With the classical umbral calculus notation (see e.g. [12]), define $(c_l + \widehat{c}_m)^n$ and $(\widehat{c}_l + \widehat{c}_m)^n$ for $l, m, n \geq 0$ by

$$(c_l + c_m)^n := \sum_{j=0}^{n} \binom{n}{j} c_{l+j} c_{m+n-j}\,,$$

$$(\widehat{c}_l + \widehat{c}_m)^n := \sum_{j=0}^{n} \binom{n}{j} \widehat{c}_{l+j} \widehat{c}_{m+n-j}\,,$$

respectively. The analogous concept for the Bernoulli numbers B_n, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi)\,,$$

has been extensively studied by many authors, including Agoh and Dilcher ([1, 2, 5] and references there). Define

$$(B_l + B_m)^n := \sum_{j=0}^{n} \binom{n}{j} B_{l+j} B_{m+n-j}\,.$$

Then Euler's famous formula can be written as

$$(B_0+B_0)^n=-nB_{n-1}-(n-1)B_n \quad (n\ge 1)\,. \tag{2}$$

In [1] an expression for $(B_l+B_m)^n$ was found. In addition, some initial cases were listed, including e.g.,

$$\begin{aligned}(B_0+B_1)^n&=-\frac{1}{2}(n+1)B_n-\frac{1}{2}nB_{n+1}\,,\\ (B_0+B_2)^n&=-\frac{1}{6}(n-1)B_n-\frac{1}{2}nB_{n+1}-\frac{1}{3}nB_{n+2}\,,\\ (B_1+B_1)^n&=\frac{1}{6}(n-1)B_n-B_{n+1}-\frac{1}{6}(n+3)B_{n+2}\,.\end{aligned}$$

On the other hand, an analogous formula to (2) is given by

$$(c_0+c_0)^n=-n(n-2)c_{n-1}-(n-1)c_n \quad (n\ge 0) \tag{3}$$

([13]). A different direction of extended convolution identities for Cauchy numbers is discussed in [10].

In this paper, we give an explicit formula for $(c_l+c_m)^n$ and $(\widehat{c}_l+\widehat{c}_m)^n$ $(n\ge 0)$ together with some initial cases, including

$$(c_0+c_1)^n=-\frac{1}{2}(n+1)(n-1)c_n-\frac{1}{2}nc_{n+1}\,, \tag{4}$$

$$(c_0+c_2)^n=\frac{n!}{6}\sum_{k=0}^n\frac{(-1)^{n-k}(k-1)c_k}{k!}-\frac{1}{6}n(2n+1)c_{n+1}-\frac{1}{3}nc_{n+2}\,, \tag{5}$$

$$(c_1+c_1)^n=-\frac{n!}{6}\sum_{k=0}^n\frac{(-1)^{n-k}(k-1)c_k}{k!}-\frac{1}{6}n(n+5)c_{n+1}-\frac{1}{6}(n+3)c_{n+2} \tag{6}$$

and

$$(\widehat{c}_0+\widehat{c}_0)^n = n!\sum_{k=0}^{n}(-1)^{n-k}\frac{\widehat{c}_k}{k!} - n\widehat{c}_n\,, \tag{7}$$

$$(\widehat{c}_0+\widehat{c}_1)^n = -\frac{(n+1)!}{2}\sum_{k=0}^{n}(-1)^{n-k}\frac{\widehat{c}_k}{k!} - \frac{1}{2}n\widehat{c}_{n+1}\,, \tag{8}$$

$$\begin{aligned}(\widehat{c}_0+\widehat{c}_2)^n &= \frac{n!}{12}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}\big(2k(n+2k-2)+5(n-k+2)(n-k+1)\big)\widehat{c}_k\\ &\quad -\frac{n}{3}\widehat{c}_{n+2}-\frac{n}{2}\widehat{c}_{n+1}\,,\end{aligned} \tag{9}$$

$$\begin{aligned}(\widehat{c}_1+\widehat{c}_1)^n &= \frac{n!}{12}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}\big((n+1)(n+2)+k(8n-9k+19)\big)\widehat{c}_k\\ &\quad -\widehat{c}_{n+1}-\frac{n+3}{6}\widehat{c}_{n+2}\,.\end{aligned} \tag{10}$$

2 Fundamental results

Euler's identity (2) is an easy consequence of the formula

$$b(x)^2 = (1-x)b(x) - xb'(x)\,,$$

where $b(x) = x/(e^x-1)$ (see e.g. [6]). Similarly, the identity (3) is obtained because $c(x) = x/\ln(1+x)$ satisfies the formula

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x)\,. \tag{11}$$

From the generating function for the c_n we obtain for $i, \nu \geq 0$,

$$x^i c^{(\nu)}(x) = \sum_{n=0}^{\infty}\frac{n!}{(n-i)!}c_{n+\nu-i}\frac{x^n}{n!}\,. \tag{12}$$

Therefore the identity (11) immediately leads to the formula (3). Differentiating both sides of (11) with respect to x and dividing them by 2, we obtain

$$c(x)c'(x) = -\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\,. \tag{13}$$

Then by applying (12), we have the identity (4).

In general, by differentiating both sides of (11) μ times with respect to x, we have the following. The left-hand side is due to the General Leibniz's rule. The right-hand side can be proved by induction.

Proposition 1. *For* $\mu \ge 0$

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}c^{(\kappa)}(x)c^{(\mu-\kappa)}(x)$$
$$= -(\mu-2)\mu c^{(\mu-1)}(x) - \big((2\mu-1)x+\mu-1\big)c^{(\mu)}(x) - x(1+x)c^{(\mu+1)}(x)\,. \tag{14}$$

By applying (12) in Proposition 1, we obtain the following result.

Theorem 1. *For* $\mu \ge 0$ *and* $n \ge 0$ *we have*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(c_\kappa + c_{\mu-\kappa})^n = -(n+\mu-2)(n+\mu)c_{n+\mu-1} - (n+\mu-1)c_{n+\mu}\,.$$

Examples. If we put $\mu = 0, 1$ in Theorem 1, we have the identities (3) and (4), respectively. If we put $\mu = 2, 3, 4$ in Theorem 1, we obtain

$$(c_0+c_2)^n + (c_1+c_1)^n = -\frac{1}{2}n(n+2)c_{n+1} - \frac{1}{2}(n+1)c_{n+2}\,, \tag{15}$$
$$(c_0+c_3)^n + 3(c_1+c_2)^n = -\frac{1}{2}(n+1)(n+3)c_{n+2} - \frac{1}{2}(n+2)c_{n+3}\,, \tag{16}$$
$$(c_0+c_4)^n + 4(c_1+c_3)^n + 3(c_2+c_2)^n$$
$$= -\frac{1}{2}(n+2)(n+4)c_{n+3} - \frac{1}{2}(n+3)c_{n+4}\,. \tag{17}$$

Since $\widehat{c}(x) = x/(1+x)\ln(1+x)$ satisfies the formula

$$\widehat{c}(x)^2 = -x\widehat{c}'(x) + \frac{1}{1+x}\widehat{c}(x)\,, \tag{18}$$

by $1/(1+x) = \sum_{i=0}^{\infty}(-1)^i x^i$ and the fact

$$x^i\widehat{c}^{(\nu)}(x) = \sum_{n=0}^{\infty}\frac{n!}{(n-i)!}\widehat{c}_{n+\nu-i}\frac{x^n}{n!} \quad (i, \nu \ge 0)\,, \tag{19}$$

we have

$$(\widehat{c}_0+\widehat{c}_0)^n = -n\widehat{c}_n + n!\sum_{k=0}^{n}(-1)^{n-k}\frac{\widehat{c}_k}{k!} \quad (n \geq 0)\,,$$

which is the formula (7). Differentiating both sides of (18) by x and dividing them by 2, we obtain

$$\widehat{c}(x)\widehat{c}'(x) = -\frac{1}{2}x\widehat{c}''(x) - \frac{x}{2(1+x)}\widehat{c}'(x) - \frac{1}{2(1+x)^2}\widehat{c}(x)\,. \tag{20}$$

Then by (19) and for $k \geq 1$

$$\frac{1}{(1+x)^k} = \sum_{i=0}^{\infty}(-1)^i\binom{k+i-1}{i}x^i\,, \tag{21}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\widehat{c}_0+\widehat{c}_1)^n\frac{x^n}{n!} &= -\frac{1}{2}\sum_{n=0}^{\infty}n\widehat{c}_{n+1}\frac{x^n}{n!} - \frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\sum_{n=0}^{\infty}\frac{n!}{(n-i-1)!}\widehat{c}_{n-i}\frac{x^n}{n!}\\ &\quad -\frac{1}{2}\sum_{i=0}^{\infty}(-1)^i(i+1)\sum_{n=0}^{\infty}\frac{n!}{(n-i)!}\widehat{c}_{n-i}\frac{x^n}{n!}\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}n\widehat{c}_{n+1}\frac{x^n}{n!} - \frac{1}{2}\sum_{i=0}^{n}(-1)^i\sum_{n=0}^{\infty}\frac{(n+1)!}{(n-i)!}\widehat{c}_{n-i}\frac{x^n}{n!}\,,\end{aligned}$$

yielding the formula (8).

In general, by differentiating both sides of (18) by x at μ times, we have the following.

Proposition 2. *For $\mu \geq 0$*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\widehat{c}^{(\kappa)}(x)\widehat{c}^{(\mu-\kappa)}(x) = \sum_{\nu=0}^{\mu}(-1)^{\mu-\nu}\frac{\mu!\widehat{c}^{(\nu)}(x)}{\nu!(1+x)^{\mu-\nu+1}} - \mu\widehat{c}^{(\mu)}(x) - x\widehat{c}^{(\mu+1)}(x)\,.$$

By applying (21) and (19) in Proposition 2, we obtain the following result.

Theorem 2. *For $\mu \geq 0$ and $n \geq 0$ we have*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(\widehat{c}_\kappa+\widehat{c}_{\mu-\kappa})^n = -(n+\mu)\widehat{c}_{n+\mu} + \sum_{\nu=0}^{\mu}\frac{\mu!n!}{\nu!}\sum_{i=0}^{n}(-1)^{\mu-\nu+i}\binom{\mu-\nu+i}{i}\frac{\widehat{c}_{n+\nu-i}}{(n-i)!}\,.$$

Examples. If we put $\mu = 0, 1$ in Theorem 2, we have the identities (7) and (8), respectively. If we put $\mu = 2, 3, 4$ in Theorem 2, we obtain

$$(\widehat{c}_0 + \widehat{c}_2)^n + (\widehat{c}_1 + \widehat{c}_1)^n = \frac{(n+2)!}{2} \sum_{k=0}^{n} (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{n+2}{2} \widehat{c}_{n+1} - \frac{n+1}{2} \widehat{c}_{n+2}\,, \tag{22}$$

$$\begin{aligned}&(\widehat{c}_0 + \widehat{c}_3)^n + 3(\widehat{c}_1 + \widehat{c}_2)^n \\ &= -\frac{(n+3)!}{2} \sum_{k=0}^{n} (-1)^{n-k} \frac{\widehat{c}_k}{k!} + \frac{(n+2)(n+3)}{2} \widehat{c}_{n+1} - \frac{n+3}{2} \widehat{c}_{n+2} - \frac{n+2}{2} \widehat{c}_{n+3}\,,\end{aligned} \tag{23}$$

$$\begin{aligned}&(\widehat{c}_0 + \widehat{c}_4)^n + 4(\widehat{c}_1 + \widehat{c}_3)^n + 3(\widehat{c}_2 + \widehat{c}_2)^n \\ &= \frac{(n+4)!}{2} \sum_{k=0}^{n} (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{(n+2)(n+3)(n+4)}{2} \widehat{c}_{n+1} + \frac{(n+3)(n+4)}{2} \widehat{c}_{n+2} \\ &\quad - \frac{n+4}{2} \widehat{c}_{n+3} - \frac{n+3}{2} \widehat{c}_{n+4}\,.\end{aligned} \tag{24}$$

Next, notice that $c(x)$ satisfies the identity

$$3c(x)c''(x) = -(1+x)xc^{(3)}(x) - \frac{3}{2}xc''(x) + \frac{x}{2(1+x)}c'(x) - \frac{1}{2(1+x)}c(x)\,.$$

Since by (12) we have

$$x^2 c^{(3)}(x) = \sum_{n=0}^{\infty} n(n-1) c_{n+1} \frac{x^n}{n!}\,,$$

$$x c^{(3)}(x) = \sum_{n=0}^{\infty} n c_{n+2} \frac{x^n}{n!}\,,$$

$$x c''(x) = \sum_{n=0}^{\infty} n c_{n+1} \frac{x^n}{n!}$$

and

$$\begin{aligned}\frac{x}{1+x}c'(x) &= \left(\sum_{\nu=0}^{\infty} (-x)^\nu\right) \left(\sum_{n=0}^{\infty} n c_n \frac{x^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{(-1)^{n-k} k c_k}{k!} \right) \frac{x^n}{n!}\end{aligned}$$

and

$$\frac{1}{1+x}c(x) = \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{n}\frac{(-1)^{n-k}c_k}{k!}\right)\frac{x^n}{n!},$$

we get

$$3(c_0+c_2)^n = -nc_{n+2} - n\left(n+\frac{1}{2}\right)c_{n+1} + \frac{n!}{2}\sum_{k=0}^{n}\frac{(-1)^{n-k}(k-1)c_k}{k!},$$

yielding the identity (5). In addition, by the identities (5) and (15), we have

$$(c_1+c_1)^n = -\frac{n!}{6}\sum_{k=0}^{n}\frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2},$$

which is the identity (6).

Next, notice that $\widehat{c}(x)$ satisfies the identity

$$\widehat{c}(x)\widehat{c}''(x) = -\frac{x}{3}\widehat{c}^{(3)}(x) - \frac{x}{2(1+x)}\widehat{c}''(x) + \frac{x}{6(1+x)^2}\widehat{c}'(x) + \frac{5}{6(1+x)^3}\widehat{c}(x).$$

By (19) and (21)

$$\begin{aligned}
&(\widehat{c}_0+\widehat{c}_2)^n \\
&= -\frac{n}{3}\widehat{c}_{n+2} - \frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\frac{n!}{(n-i-1)!}\widehat{c}_{n-i+1} \\
&\quad + \frac{1}{6}\sum_{i=0}^{\infty}(-1)^i\frac{(i+1)n!}{(n-i-1)!}\widehat{c}_{n-i} + \frac{5}{6}\sum_{i=0}^{\infty}(-1)^i\frac{(i+2)(i+1)}{2}\frac{n!}{(n-i)!}\widehat{c}_{n-i} \\
&= -\frac{n}{3}\widehat{c}_{n+2} - \frac{n}{2}\widehat{c}_{n+1} + \frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{(n-i-1)!}\widehat{c}_{n-i} \\
&\quad + \frac{n!}{6}\sum_{i=0}^{n-1}\frac{(-1)^i(i+1)}{(n-i-1)!}\widehat{c}_{n-i} + \frac{5n!}{12}\sum_{i=0}^{n-2}\frac{(-1)^i(i+2)(i+1)}{(n-i)!}\widehat{c}_{n-i} \\
&= -\frac{n}{3}\widehat{c}_{n+2} - \frac{n}{2}\widehat{c}_{n+1} \\
&\quad + \frac{n!}{12}\sum_{i=0}^{n}\frac{(-1)^i}{(n-i)!}\big(6(n-i)(n-i-1) + 2(i+1)(n-i) + 5(i+2)(i+1)\big)\widehat{c}_{n-i},
\end{aligned}$$

yielding the identity (9). In addition, by the identities (9) and (22), we have

$$(\widehat{c}_1+\widehat{c}_1)^n = \frac{n!}{12}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}\big((n+1)(n+2)+k(8n-9k+19)\big)\widehat{c}_k-\widehat{c}_{n+1}-\frac{n+3}{6}\widehat{c}_{n+2}\,,$$

which is the identity (10).

Proposition 3. *For $m \geq 0$ we have*

$$\widehat{c}(x)\widehat{c}^{(m)}(x)$$
$$= -\frac{x}{m+1}\widehat{c}^{(m+1)}(x) - \sum_{k=0}^{m-1}\binom{m}{k}\frac{c_{k+1}}{k+1}\frac{x}{(1+x)^{k+1}}\widehat{c}^{(m-k)}(x) + \frac{\widehat{c}_m}{(1+x)^{m+1}}\widehat{c}(x)\,. \tag{25}$$

Lemma 1. *For $n \geq 0$ we have*

$$\widehat{c}^{(n)}(x) = \left(\frac{-1}{1+x}\right)^{n+1}\sum_{j=0}^{n}\frac{\widehat{g}_{n,j}}{\big(\ln(1+x)\big)^{j+1}}\,,$$

where

$$\widehat{g}_{n,j} = j!\left(n\begin{bmatrix} n \\ j+1\end{bmatrix} - \begin{bmatrix} n \\ j\end{bmatrix}x\right)\,.$$

Proof. Since $\widehat{g}_{1,0} = 1$ and $\widehat{g}_{1,1} = -x$, for $n = 1$ we have

$$\begin{aligned}\frac{d}{dx}\frac{x}{(1+x)\ln(1+x)} &= \frac{\ln(1+x)-x}{(1+x)^2\big(\ln(1+x)\big)^2}\\ &= \frac{1}{(1+x)^2}\left(\frac{\widehat{g}_{1,0}}{\ln(1+x)} + \frac{\widehat{g}_{1,1}}{\big(\ln(1+x)\big)^2}\right)\,.\end{aligned}$$

Assume that the result holds for n. Then

$$\begin{aligned}
&\frac{d^{n+1}}{dx^{n+1}}\frac{x}{(1+x)\ln(1+x)}\\
&=(n+1)\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{\widehat{g}_{n,j}}{\big(\ln(1+x)\big)^{j+1}}\\
&\quad+\left(\frac{-1}{1+x}\right)^{n+1}\sum_{j=0}^{n}\frac{-j!\left[{n \atop j}\right]\big(\ln(1+x)\big)^{j+1}-\widehat{g}_{n,j}(j+1)\big(\ln(1+x)\big)^{j}\frac{1}{1+x}}{\big(\ln(1+x)\big)^{2j+2}}\\
&=n\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{\widehat{g}_{n,j}}{\big(\ln(1+x)\big)^{j+1}}+\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{\widehat{g}_{n,j}}{\big(\ln(1+x)\big)^{j+1}}\\
&\quad+\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{j!\left[{n \atop j}\right](1+x)}{\big(\ln(1+x)\big)^{j+1}}+\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{\widehat{g}_{n,j}(j+1)}{\big(\ln(1+x)\big)^{j+2}}\\
&=\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n}\frac{n\widehat{g}_{n,j}+j!\left[{n+1 \atop j+1}\right]}{\big(\ln(1+x)\big)^{j+1}}+\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=1}^{n+1}\frac{j\widehat{g}_{n,j-1}}{\big(\ln(1+x)\big)^{j+1}}\\
&=\left(\frac{-1}{1+x}\right)^{n+2}\sum_{j=0}^{n+1}\frac{\widehat{g}_{n+1,j}}{\big(\ln(1+x)\big)^{j+1}}\,.
\end{aligned}$$

□

Lemma 2. *For integers m and j with $m\ge j\ge 0$ we have*

$$\widehat{g}_{m+1,j}-(m+1)\widehat{g}_{m,j-1}=\sum_{k=0}^{m-j}\binom{m+1}{k+1}(-1)^k c_{k+1}\widehat{g}_{m-k,j}\,.$$

Proof. It is enough to prove

$$\left[{m+1 \atop j+1}\right]-\frac{m}{j}\left[{m \atop j}\right]=\sum_{k=0}^{m-j}\binom{m+1}{k+1}(-1)^k c_{k+1}\frac{m-k}{m+1}\left[{m-k \atop j+1}\right] \tag{26}$$

and

$$\left[{m+1 \atop j}\right]-\frac{m+1}{j}\left[{m \atop j-1}\right]=\sum_{k=0}^{m-j}\binom{m+1}{k+1}(-1)^k c_{k+1}\left[{m-k \atop j}\right]. \tag{27}$$

The identity (26) holds because by (1) formulae in Table 241 in [7]

$$
\begin{aligned}
&\sum_{k=0}^{m-j+1}\binom{m+1}{k}(-1)^k c_k \frac{m-k+1}{m+1}\left[{m-k+1 \atop j+1}\right]\\
&=\sum_{k=0}^{m-j+1}\binom{m+1}{k}(-1)^k\sum_{i=0}^{k}\frac{(-1)^{k-i}}{i+1}\left[{k \atop i}\right]\frac{m-k+1}{m+1}\left[{m-k+1 \atop j+1}\right]\\
&=\sum_{i=0}^{m-j+1}\frac{(-1)^i}{i+1}\sum_{k=i}^{m-j+1}\binom{m+1}{k}\left[{k \atop i}\right]\frac{m-k+1}{m+1}\left[{m-k+1 \atop j+1}\right]\\
&=\sum_{i=0}^{m-j+1}\frac{(-1)^i}{i+1}\binom{i+j}{j}\left[{m+1 \atop i+j+1}\right]\\
&=\frac{m}{j}\left[{m \atop j}\right].
\end{aligned}
$$

Similarly, the identity (27) holds because

$$
\begin{aligned}
&\sum_{k=0}^{m-j+1}\binom{m+1}{k}(-1)^k c_k \left[{m-k+1 \atop j}\right]\\
&=\sum_{k=0}^{m-j+1}\binom{m+1}{k}(-1)^k\sum_{i=0}^{k}\frac{(-1)^{k-i}}{i+1}\left[{k \atop i}\right]\left[{m-k+1 \atop j}\right]\\
&=\sum_{i=0}^{m-j+1}\frac{(-1)^i}{i+1}\sum_{k=i}^{m-j+1}\binom{m+1}{k}\left[{k \atop i}\right]\left[{m-k+1 \atop j}\right]\\
&=\sum_{i=0}^{m-j+1}\frac{(-1)^i}{i+1}\binom{i+j}{j}\left[{m+1 \atop i+j}\right]\\
&=\frac{m+1}{j}\left[{m \atop j-1}\right].
\end{aligned}
$$

□

We also need the following relation ([11, Theorem 2.4 (2.1)]) in order to prove Proposition 3.

Lemma 3. *For* $m \ge 0$

$$(-1)^m\frac{\widehat{c}_m}{m!}=\sum_{k=0}^{m}(-1)^k\frac{c_k}{k!}.$$

Proof of Proposition 3. The identity (25) holds for $m=0$ and $m=1$ because of (18) and (20), respectively. Let $m \ge 2$. By Lemma 1 with $\widehat{g}_{n,n} = -n!x$ ($n \ge 0$), $\widehat{g}_{n,0} = n!$ ($n \ge 1$) and $\widehat{g}_{0,0} = -x$ together with Lemmata 2 and 3

$$
\begin{aligned}
&\widehat{c}(x)\widehat{c}^{(m)}(x) + \frac{x}{m+1}\widehat{c}^{(m+1)}(x)\\
&= \frac{x}{m+1}\left(\frac{-1}{1+x}\right)^{m+2}\left(\sum_{j=0}^{m-1}\frac{\widehat{g}_{m+1,j+1}-(m+1)\widehat{g}_{m,j}}{(\ln(1+x))^{j+2}} + \frac{(m+1)!}{\ln(1+x)}\right)\\
&= \frac{x}{m+1}\left(\frac{-1}{1+x}\right)^{m+2}\left(\sum_{j=0}^{m-1}\frac{1}{\left(\ln(1+x)\right)^{j+2}}\sum_{k=0}^{m-j-1}\binom{m+1}{k+1}(-1)^k c_{k+1}\widehat{g}_{m-k,j+1}\right.\\
&\quad\left. + \frac{(m+1)!}{\ln(1+x)}\sum_{k=0}^{m-1}(-1)^k\frac{c_{k+1}}{(k+1)!} + \frac{(-1)^m(m+1)\widehat{c}_m}{\ln(1+x)}\right)\\
&= \frac{x}{m+1}\left(\frac{-1}{1+x}\right)^{m+2}\left(\sum_{j=0}^{m}\frac{m+1}{\left(\ln(1+x)\right)^{j+1}}\sum_{k=0}^{m-j}\frac{c_{k+1}}{k+1}\binom{m}{k}(-1)^k\widehat{g}_{m-k,j}\right.\\
&\quad\left. + \frac{(-1)^m}{\ln(1+x)}\left(c_{m+1}x+(m+1)\widehat{c}_m\right)\right)\\
&= \left(\frac{-1}{1+x}\right)^{m+2}\sum_{k=0}^{m}\binom{m}{k}\frac{c_{k+1}}{k+1}(-1)^k\sum_{j=0}^{m-k}\frac{x\widehat{g}_{m-k,j}}{\left(\ln(1+x)\right)^{j+1}}\\
&\quad + \left(\frac{-1}{1+x}\right)^{m+2}\frac{x}{\ln(1+x)}(-1)^m\left(\frac{c_{m+1}}{m+1}x+\widehat{c}_m\right)\\
&= -\sum_{k=0}^{m-1}\binom{m}{k}\frac{c_{k+1}}{k+1}\frac{x}{(1+x)^{k+1}}\widehat{c}^{(m-k)}(x) + \frac{\widehat{c}_m}{(1+x)^{m+1}}\widehat{c}(x)\,.
\end{aligned}
$$

$\square$

By Proposition 3, we obtain the following theorem.

Theorem 3. *For $m \ge 0$ and $n \ge 0$ we have*

$$
\begin{aligned}
&(\widehat{c}_0 + \widehat{c}_m)^n \\
&= -\frac{n}{m+1}\widehat{c}_{n+m} \\
&\quad - \frac{1}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k+1}c_{k+1}\sum_{i=0}^{n-1}(-1)^i\binom{k+i}{i}\frac{n!}{(n-i-1)!}\widehat{c}_{n+m-k-i-1} \\
&\quad + \widehat{c}_m\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\frac{n!}{(n-i)!}\widehat{c}_{n-i}\,.
\end{aligned}
$$

Examples. When $m = 0$, $m = 1$ and $m = 2$, we have the formulae (7), (8) and (9), respectively. If $m = 3$, we have

$$
\begin{aligned}
(\widehat{c}_0 + \widehat{c}_3)^n = &-\frac{(n+1)!}{8}\sum_{k=0}^{n}\frac{(-1)^{n-k}\big(2(2n^2-6n+9)-(n-k)(n+9k-27)\big)\widehat{c}_k}{k!} \\
&+\frac{n(2n-1)\widehat{c}_{n+1}}{4}-\frac{n\widehat{c}_{n+2}}{2}-\frac{n\widehat{c}_{n+3}}{4}\,.
\end{aligned}
$$

Proof of Theorem 3. By (19)

$$
x\widehat{c}^{(m+1)}(x) = \sum_{n=0}^{\infty} n\widehat{c}_{n+m}\frac{x^n}{n!}\,.
$$

By (21) we get

$$
\begin{aligned}
&\sum_{k=0}^{m-1}\binom{m+1}{k+1}\frac{c_{k+1}x}{(1+x)^{k+1}}\widehat{c}^{(m-k)}(x) \\
&= \sum_{k=0}^{m-1}\binom{m+1}{k+1}c_{k+1}\sum_{i=0}^{\infty}(-1)^i\binom{k+i}{i}x^{i+1}\widehat{c}^{(m-k)}(x) \\
&= \sum_{k=0}^{m-1}\binom{m+1}{k+1}c_{k+1}\sum_{i=0}^{n-1}(-1)^i\binom{k+i}{i}\sum_{n=0}^{\infty}\frac{n!}{(n-i-1)!}\widehat{c}_{n+m-k-i-1}\frac{x^n}{n!}\,.
\end{aligned}
$$

Finally, for $m \ge 0$

$$
\frac{\widehat{c}_m}{(1+x)^{m+1}}\widehat{c}(x) = \widehat{c}_m\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{n=0}^{\infty}\frac{n!}{(n-i)!}\widehat{c}_{n-i}\frac{x^n}{n!}\,.
$$

□

Similarly to Theorem 3 for Cauchy numbers of the second kind, we have the following for Cauchy numbers of the first kind.

Theorem 4. *For $m \ge 2$ and $n \ge 0$ we have*

$$
\begin{aligned}
&(c_0 + c_m)^n = \\
&- \frac{n}{m+1}\left(\frac{2n+m-1}{2} c_{n+m-1} + c_{n+m}\right) \\
&- \sum_{k=1}^{m-1} \frac{c_{k+1}}{k+1}\binom{m}{k} \sum_{i=0}^{n-1}(-1)^i \binom{k+i-1}{i} \frac{n!}{(n-i-1)!} c_{n+m-k-i-1} \\
&+ c_m \sum_{i=0}^{n}(-1)^i \binom{m+i-2}{i} \frac{n!}{(n-i)!} c_{n-i}\,.
\end{aligned}
$$

The expression of $(c_0 + c_m)^n$ is based upon the following relation.

Proposition 4. *For $m \ge 0$ we have*

$$
\begin{aligned}
&c(x)c^{(m)}(x) \\
&\quad = -\frac{x(1+x)}{m+1} c^{(m+1)}(x) - \sum_{k=0}^{m-1} \frac{c_{k+1}}{k+1}\binom{m}{k} \frac{x}{(1+x)^k} c^{(m-k)}(x) \\
&\qquad\qquad + \frac{c_m}{(1+x)^{m-1}} c(x)\,.
\end{aligned}
$$

We need the following Lemma.

Lemma 4. *For $n \ge 0$ we have*

$$
c^{(n)}(x) = \left(\frac{-1}{1+x}\right)^n \sum_{j=0}^{n} \frac{g_{n,j}}{(\ln(1+x))^{j+1}}\,,
$$

where

$$
g_{n,j} = j!\left(x \begin{bmatrix} n \\ j \end{bmatrix} - (x+1)n \begin{bmatrix} n-1 \\ j \end{bmatrix}\right).
$$

Proof. It is proven inductively that

$$
\frac{d^n}{dx^n}\frac{1}{\ln(1+x)} = \frac{(-1)^n}{(1+x)^n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{j!}{(\ln(1+x))^{j+1}} \quad (n \ge 0)
$$

(*Cf.* [3, Lemma 4.1]). Using Leibniz's rule, we have

$$
\begin{aligned}
\frac{d^n}{dx^n}\frac{x}{\ln(1+x)} &= x\frac{d^n}{dx^n}\frac{1}{\ln(1+x)} + n\frac{d^{n-1}}{dx^{n-1}}\frac{1}{\ln(1+x)} \\
&= x\frac{(-1)^n}{(1+x)^n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\frac{j!}{(\ln(1+x))^{j+1}} \\
&\quad + n\frac{(-1)^{n-1}}{(1+x)^{n-1}}\sum_{j=0}^{n-1}\begin{bmatrix} n-1 \\ j \end{bmatrix}\frac{j!}{(\ln(1+x))^{j+1}} \\
&= \left(\frac{-1}{1+x}\right)^n\sum_{j=0}^{n}\frac{j!}{(\ln(1+x))^{j+1}}\left(x\begin{bmatrix} n \\ j \end{bmatrix} - (x+1)n\begin{bmatrix} n-1 \\ j \end{bmatrix}\right).
\end{aligned}
$$

□

3 Main result

We can show the following proposition in order to obtain the main result.

Proposition 5. *Let l, m be nonnegative integers with $m \ge l \ge 1$. Then*

$$
\begin{aligned}
c^{(l)}(x)c^{(m)}(x) &= -\frac{l!m!}{(l+m+1)!}x(1+x)c^{(l+m+1)}(x) - \frac{l!m!}{(l+m)!}(2x+1)c^{(l+m)}(x) \\
&- \frac{l!m!}{(l+m-1)!}c^{(l+m-1)}(x) + \sum_{k=0}^{m-l-1}\frac{a_{l,m.m-k}x + b_{l,m.m-k}}{(1+x)^{l+k}}c^{(m-k)}(x) \\
&+ \sum_{k=0}^{l-2}\frac{a_{l,m,l-k}x + b_{l,m,l-k}}{(1+x)^{m+k}}c^{(l-k)}(x) + \frac{a_{l,m,1}x}{(1+x)^{l+m-1}}c'(x) + \frac{b_{l,m,0}}{(1+x)^{l+m-1}}c(x),
\end{aligned}
$$

where for $l+1 \le r = m-k \le m$

$$
\begin{aligned}
a_{l,m,r} &= (-1)^{l+1}\binom{m}{k}\sum_{i=0}^{l} i!\binom{l}{i}\binom{l+k-2}{i}\frac{c_{l+k-i+1}}{l+k-i+1}, \\
b_{l,m,r} &= (-1)^{l} l\binom{m}{k}\sum_{i=0}^{l-1} i!\binom{l-1}{i}\binom{l+k-2}{i}\frac{c_{l+k-i}}{l+k-i},
\end{aligned}
$$

and for $1 \le r = l - k \le l$

$$a_{l,m,r} = \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1}$$
$$\times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-2}{i} c_{l+m-i-r+1}$$
$$+ \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$

except

$$a_{l,m,3} = -\frac{m(m-1)}{3!} \sum_{i=0}^{4} i! \binom{4}{i} \binom{m-1}{i} c_{m-i+2}$$
$$+ \frac{4(m-3)}{3!} \sum_{i=0}^{3} i! \binom{3}{i} \binom{m}{i} c_{m-i+2} \quad (l = 4)$$

and

$$a_{l,m,2} = (-1)^{l+1} \frac{m-l}{2} \sum_{i=0}^{l} i! \binom{l}{i} \binom{l+m-3}{i} c_{l+m-i-1} \quad (l = 2, 3)\,,$$

and for $2 \le r = l - k \le l$

$$b_{l,m,r} = \sum_{j=1}^{r-1} (-1)^{l+j+1} \binom{l}{j} \binom{m}{r-j} \binom{r}{j}^{-1} \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-2}{i} c_{l+m-i-r}$$

except

$$b_{l,m,0} = (-1)^{l} \sum_{i=0}^{l} i! \binom{l}{i} \binom{l+m-2}{i} c_{l+m-i} = -a_{l,m,1}\,.$$

With this proposition we now obtain our main result.

Theorem 5. *Let l, m, n be integers with $m \geq l \geq 1$ and $n \geq 0$. Then*

$$\begin{aligned}
&(c_l + c_m)^n \\
&= -\frac{l!m!}{(l+m+1)!}\big((n+l+m+1)c_{n+l+m} + (n+l+m+1)(n+l+m)c_{n+l+m-1}\big) \\
&\quad + \sum_{k=0}^{m-l-1} \sum_{i=0}^{n} \frac{(-1)^{i-1} n!}{(n-i)!} \\
&\quad \times \left(\binom{l+k+i-2}{i-1} a_{l,m,m-k} - \binom{l+k+i-1}{i} b_{l,m,m-k} \right) c_{n+m-k-i} \\
&\quad + \sum_{k=0}^{l-2} \sum_{i=0}^{n} \frac{(-1)^{i-1} n!}{(n-i)!} \\
&\quad \times \left(\binom{m+k+i-2}{i-1} a_{l,m,l-k} - \binom{m+k+i-1}{i} b_{l,m,l-k} \right) c_{n+l-k-i} \\
&+ \sum_{i=0}^{n} (-1)^i \binom{l+m+i-2}{i} \frac{n!}{(n-i)!} (n-i-1) a_{l,m,1} c_{n-i}\,.
\end{aligned}$$

where $a_{l,m,r}$ and $b_{l,m,r}$ are defined in Proposition 5.

Similarly, for general integers l and m, we obtain the following result.

Proposition 6. *Let l, m be fixed nonnegative integers with $m \geq l \geq 1$. Then*

$$\begin{aligned}
\widehat{c}^{(l)}(x)\widehat{c}^{(m)}(x) &= -\frac{l!m!}{(l+m+1)!} x \widehat{c}^{(l+m+1)}(x) - \frac{l!m!}{(l+m)!} \widehat{c}^{(l+m)}(x) \\
&+ \sum_{k=0}^{m-l-1} \frac{a_{l,m.m-k} x + b_{l,m.m-k}}{(1+x)^{l+k+1}} \widehat{c}^{(m-k)}(x) + \sum_{k=0}^{l} \frac{a_{l,m,l-k} x + b_{l,m,l-k}}{(1+x)^{m+k+1}} \widehat{c}^{(l-k)}(x)\,,
\end{aligned}$$

where for $l+1 \leq r = m-k \leq m$

$$a_{l,m,r} = (-1)^{l+1} \binom{m}{k} \sum_{i=0}^{l} i! \binom{l}{i} \binom{l+k-1}{i} \frac{c_{l+k-i+1}}{l+k-i+1}\,,$$

$$b_{l,m,r} = (-1)^l l \binom{m}{k} \sum_{i=0}^{l-1} i! \binom{l-1}{i} \binom{l+k-1}{i} \frac{c_{l+k-i}}{l+k-i}\,,$$

and for $1 \le r = l - k \le l$

$$a_{l,m,r} = \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1}$$
$$\times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$
$$+ \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$

with $a_{l,m,0} = 0$ *and for* $0 \le r = l - k \le l$

$$b_{l,m,r} = \sum_{j=1}^{r} (-1)^{l+j+1} \binom{l}{j} \binom{m}{r-j} \binom{r}{j}^{-1} \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r}$$
$$+ (-1)^{l+r} \binom{l}{r} \sum_{i=0}^{l-r} i! \binom{l-r}{i} \binom{l+m-r}{i} \widehat{c}_{l+m-i-r} \,.$$

By using Proposition 6, we obtain explicit expressions for $(\widehat{c}_l + \widehat{c}_m)^n$.

Theorem 6. *Let* l, m, n *be integers with* $m \ge l \ge 1$ *and* $n \ge 0$. *Then*

$$(\widehat{c}_l + \widehat{c}_m)^n$$
$$= -\frac{l!m!}{(l+m+1)!}(n+l+m+1)\widehat{c}_{n+l+m}$$
$$+ \sum_{k=0}^{m-l-1} \sum_{i=0}^{n} \frac{(-1)^{i-1} n!}{(n-i)!}$$
$$\times \left(\binom{l+k+i-1}{i-1} a_{l,m,m-k} - \binom{l+k+i}{i} b_{l,m,m-k} \right) \widehat{c}_{n+m-k-i}$$
$$+ \sum_{k=0}^{l} \sum_{i=0}^{n} \frac{(-1)^{i-1} n!}{(n-i)!}$$
$$\times \left(\binom{m+k+i-1}{i-1} a_{l,m,l-k} - \binom{m+k+i}{i} b_{l,m,l-k} \right) \widehat{c}_{n+l-k-i} \,.$$

where $a_{l,m,r}$ *and* $b_{l,m,r}$ *are defined in Proposition 6, and* $\binom{n}{-1} = 0$ $(n \ge 0)$.

References

[1] T. Agoh and K. Dilcher, *Convolution identities and lacunary recurrences for Bernoulli numbers*, J. Number Theory **124** (2007), 105–122.

[2] T. Agoh and K. Dilcher, *Higher-order recurrences for Bernoulli numbers*, J. Number Theory **129** (2009), 1837–1847.

[3] T. Agoh and K. Dilcher, *Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind*, Fibonacci Quart. **48** (2010), 4–12.

[4] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[5] K. Dilcher, *Sums of products of Bernoulli numbers*, J. Number Theory **60** (1996), 23–41.

[6] I. M. Gessel, *On Miki's identity for Bernoulli numbers*, J. Number Theory **110** (2005), 75–82.

[7] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, Reading, 1994.

[8] Ch. Jordan, *Calculus of finite differences*, and ed., Chelsea Publ. Co., New York, 1950.

[9] T. Komatsu, *Poly-Cauchy numbers*, Kyushu J. Math. **67** (2013), 143–153.

[10] T. Komatsu, *Sums of products of Cauchy numbers, including poly-Cauchy numbers*, J. Discrete Math. **2013** (2013), Article ID 373927, 10 pages.

[11] D. Merlini, R. Sprugnoli and M. C. Verri, *The Cauchy numbers*, Discrete Math. **306** (2006) 1906–1920.

[12] S. Roman, *The Umbral Calculus*, Dover, 2005.

[13] F.-Z. Zhao, *Sums of products of Cauchy numbers*, Discrete Math. **309** (2009) 3830–3842.

Graduate School of Science and Technology
Hirosaki University
Hirosaki 036-8561
JAPAN
E-mail address: komatsu@cc.hirosaki-u.ac.jp

弘前大学・大学院理工学研究科　小松　尚夫